The Characteristic Impedance of a Family of Rectangular Coaxial Structures with Off-Centered Strip Inner Conductors

HENRY J. RIBLET, FELLOW, IEEE

Abstract—**A two-parameter family of hyperelliptic integrals is exhibited which maps the upper half-plane into the interior of a rectangle pierced by a reentrant line. Since these integrals can be expressed as the sum of elliptic integrals of the first kind, the odd- and even-mode characteristic impedances of a two-parameter family of coaxial structures whose inner conductors are strips, displaced perpendicularly to their width from the center of the outer rectangular conductor, can be expressed in terms of these well-known functions. Numerical values are given for a range of values of W/B and B_1/B. Here W is the width of the strip, B is the height of the outer rectangular conductor, and B_1 is the distance from the strip to the farthest parallel wall of the outer conductor.**

I. Introduction

THE DESIGN of a broad class of coupled transmission-line directional couplers [1]–[3] depends on the even- and odd-mode characteristic impedances of rectangular coaxial line in which the rectangular inner conductor, sometimes idealized as being infinitely thin, is displaced from the center of the outer rectangular conductor in a direction parallel to one of its sides. In the only case for which an exact solution is known [4], the inner conductor is a strip which is displaced from the center of the rectangular outer conductor parallel to its width.

In this paper, the even- and odd-mode characteristic impedances of a two-parameter family[1] of coaxial structures, in which the inner conductor is a strip which is displaced from the center of the rectangular outer conductor perpendicular to its width, are given. These exact solutions not only permit the exact design of broadside-coupled strip couplers [5] but also provide solutions of known accuracy with which more general approximations may be compared.

The Schwartz–Cristoffel transformation

$$Z=\int_0^t \frac{(d-t)\,dt}{\sqrt{t(t^2-(a+1/a)t+1)(t^2-(b+1/b)t+1)}} \quad (1)$$

maps the upper half of the t-plane in Fig. 1 into the interior of the reentrant, quadrilateral polygon in the Z-plane of Fig. 1. The capital letters indicate corresponding points in the two figures. We shall see how the value of d between $1/b$ and $1/a$ can be chosen so that sides OF and AB are the same length. It is then clear that, by reflecting the Z-plane polygon in the line FB, a rectangular coaxial structure is formed in which the outer conductor consists of a rectangle of height OA and width $2FO$ while the inner conductor is a strip of zero thickness and of width $2ED$. Although this strip is centered in the $2FO$ dimension, it is not, in general, centered in the OA dimension.

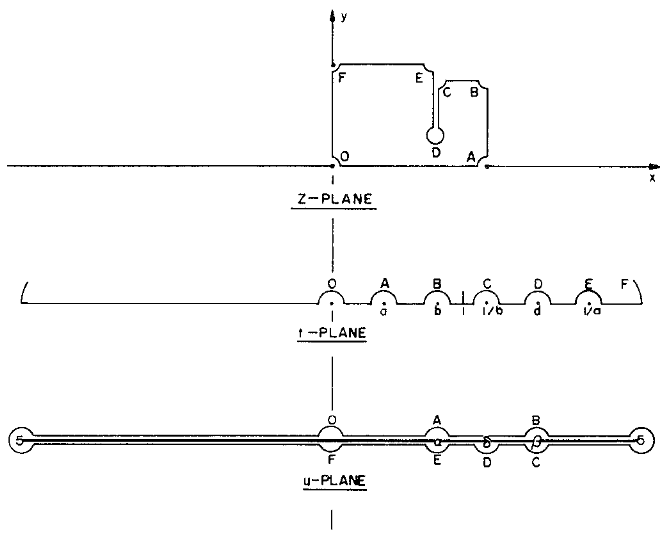

Fig. 1. Z-, t-, and u-planes.

II. Evaluation of the Integral

Since $d-t=(d-1)(1+t)/2+(d+1)(1-t)/2$,

$$Z=\frac{1}{2}\cdot\int_0^t\left\{\frac{(d-1)(1-t^2)\,dt}{\sqrt{t(1-t)^2(t^2-(a+1/a)t+1)(t^2-(b+1/b)t+1)}}\right.$$
$$\left.+\frac{(d+1)(1-t^2)\,dt}{\sqrt{t(1+t)^2(t^2-(a+1/a)t+1)(t^2-(b+1/b)t+1)}}\right\}. \quad (2)$$

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The author is with Microwave Development Laboratories, Inc., Needham Heights, MA 02194.

[1]It should be observed that this does not include all coaxial structures of this form.

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This can be written

$$Z=\frac{1}{2}\int_0^t\left[\frac{d-1}{\sqrt{\frac{(1-t)^2}{1+t^2}}}+\frac{d+1}{\sqrt{\frac{(1+t)^2}{1+t^2}}}\right]\cdot\frac{\frac{(1-t^2)\,dt}{(1+t^2)^2}}{\sqrt{\frac{t(t^2+1-(a+1/a)t)(t^2+1-(b+1/b)t)}{(1+t^2)(1+t^2)(1+t^2)}}}. \quad (3)$$

Now if the change of variable[2] $u=t/(1+t^2)$ is made, it will be found that $du/dt=(1-t^2)/(1+t^2)^2$, $1+2u=(1+t)^2/(1+t^2)$, and $1-2u=(1-t)^2/(1+t^2)$. Thus the original transformation can be written

$$Z=\frac{1}{2}\int_0^u\left(\frac{d-1}{\sqrt{1-2u}}+\frac{d+1}{\sqrt{1+2u}}\right)\cdot\frac{du}{\sqrt{u(1-(a+1/a)u)(1-(b+1/b)u)}}. \quad (4)$$

By this change of variable, the original transformation has been expressed as the sum of two elliptic integrals of the first kind. The transformation $u=t/(t^2+1)$ maps the upper half t-plane into the entire u-plane containing the two branch points ± 0.5, which may be joined by a branch cut since the points corresponding to a and $1/a$, for example, are not at the same point in the u-plane.

Thus the path of integration along the indented, real axis of the t-plane maps into the indented, closed loop surrounding the branch cut in the u-plane as shown in Fig. 1. Of course, $\alpha=a/(1+a^2)$, $\beta=b/(1+b^2)$, and $\delta=d/(1+d^2)$. It is important to observe that $a+1/a$ and $b+1/b>2$ since the geometric mean is less than the arithmetic mean. Thus α and $\beta<0.5$. Moreover, it is readily demonstrated that $\beta-\alpha>0$ whenever $1>b>\alpha>0$. Thus we know that $\alpha<\delta<\beta$.

III. The Dimensions of the Polygon

If $\gamma=0.5$, then (4), except for a scale factor, can be written

$$Z=(d-1)\int_0^u\frac{du}{\sqrt{u(\alpha-u)(\beta-u)(\gamma-u)}}+(d+1)\int_0^u\frac{du}{\sqrt{(\gamma+u)u(\alpha-u)(\beta-u)}} \quad (5)$$

where all integrations are performed on the loop encircling the branch cut in the u-plane. The values of these integrals are not single valued functions of the limits, however, but depend on the actual paths of integration. If $R_1(u)$ denotes the radical in the first integral and $R_2(u)$ denotes the radical in the second integral, the corresponding Riemann surfaces are shown in Fig. 2 with the proper branch cuts. Although at no time does the path of integration cross a branch cut in either surface, it should be noted that $R_1(u)$ and $R_2(u)$ each have different signs on opposite sides of a branch cut. With this in mind, it is readily seen that the directed lengths of the sides of the polygon are

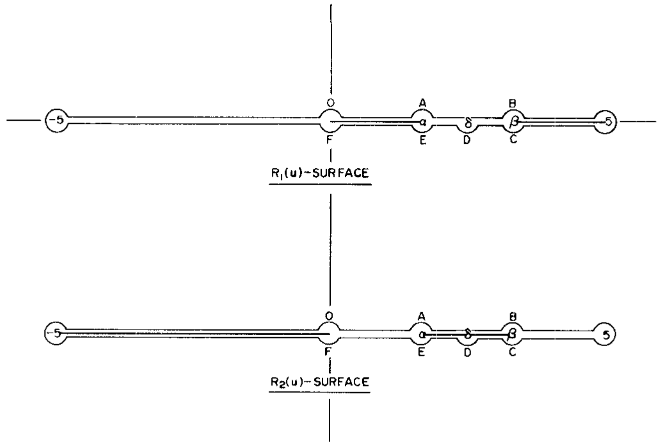

Fig. 2. $R_1(u)$ and $R_2(u)$ surfaces.

$$OA=(d-1)\int_0^\alpha\frac{du}{R_1(u)}+(d+1)\int_0^\alpha\frac{du}{R_2(u)}$$

$$AB=(d-1)\int_\alpha^\beta\frac{du}{R_1(u)}+(d+1)\int_\alpha^\beta\frac{du}{R_2(u)}$$

$$BC=2(d-1)\int_\beta^\gamma\frac{du}{R_1(u)}$$

$$CD=+(d-1)\int_\beta^\delta\frac{du}{R_1(u)}-(d+1)\int_\beta^\delta\frac{du}{R_2(u)}$$

$$DE=(d-1)\int_\delta^\alpha\frac{du}{R_1(u)}-(d+1)\int_\delta^\alpha\frac{du}{R_2(u)}$$

$$EF=+(d-1)\int_0^\alpha\frac{du}{R_1(u)}-(d+1)\int_0^\alpha\frac{du}{R_2(u)}$$

$$FO=+2(d+1)\int_{-\gamma}^0\frac{du}{R_2(u)}. \quad (6)$$

Here it is assumed that these integrals are given their principal value, that is, their value on the upper half of the u-plane.

These elliptic integrals of the first kind may be evaluated by transforming them to Legendre's normal form. This is done in the Appendix by transforming the $R_1(u)$-surface into the $R_\kappa(\omega)$-surface and $R_2(u)$-surface into the $R_\lambda(\omega)$-surface by means of linear transformations which preserve the order of the branch points. For example, the $R_\kappa(\omega)$-surface looks like the $R_1(u)$-surface except that the branch points at 0, α, β, and γ are transformed

[2]The use of this change of variable to reduce the hyperelliptic integral of the form in (2) to the sum of two elliptic integrals of the first kind is due to Roberts [6].

into the branch points $-1/\kappa$, -1, 1, and $1/\kappa$, in that order. Moreover, it is shown in the Appendix that integrals in the $R_\kappa(\omega)$-surface differ by an imaginary multiplicative constant jC_1 from the corresponding integrals in the $R_1(u)$-surface, while integrals in the $R_\lambda(\omega)$-surface differ by a real multiplicative constant C_2 from the corresponding integrals in the $R_2(u)$-surface. Hence

$$\begin{aligned}
\int_0^\alpha \frac{du}{R_1(u)} &= jC_1\int_{-1/\kappa}^{-1}\frac{d\omega}{R_\kappa(\omega)} \\
&= -jC_1\int_1^{1/\kappa}\frac{d\omega}{R_\kappa(\omega)} = C_1K'(\kappa) \\
\int_\alpha^\beta \frac{du}{R_1(u)} &= jC_1\int_{-1}^{+1}\frac{d\omega}{R_\kappa(\omega)} = j2C_1K(\kappa) \\
\int_\beta^\gamma \frac{du}{R_1(u)} &= jC_1\int_1^{1/\kappa}\frac{d\omega}{R_\kappa(\omega)} = -C_1K'(\kappa) \\
\int_{-\gamma}^0 \frac{du}{R_2(u)} &= C_2\int_{-1/\lambda}^{-1}\frac{d\omega}{R_\lambda(\omega)} \\
&= -C_2\int_1^{1/\lambda}\frac{d\omega}{R_\lambda(\omega)} = -jC_2K'(\lambda) \\
\int_0^\alpha \frac{du}{R_2(u)} &= C_2\int_{-1}^{+1}\frac{d\omega}{R_\lambda(\omega)} = 2C_2K(\lambda) \\
\int_\alpha^\beta \frac{du}{R_2(u)} &= C_2\int_1^{1/\lambda}\frac{d\omega}{R_\lambda(\omega)} = jC_2K'(\lambda).
\end{aligned} \tag{7}$$

Substituting these results in (6)

$$\begin{aligned}
OA &= (d-1)C_1K'(\kappa)+2(d+1)C_2K(\lambda) \\
AB &= j\{2(d-1)C_1K(\kappa)+(d+1)C_2K'(\lambda)\} \\
BC &= -2(d-1)C_1K'(\kappa) \\
CE &= j\{-2(d-1)C_1K(\kappa)+(d+1)C_2K'(\lambda)\} \\
EF &= (d-1)C_1K'(\kappa)-2(d+1)C_2K(\lambda) \\
FO &= -j2(d+1)C_2K'(\lambda)
\end{aligned} \tag{8}$$

where, as will be shown,

$$C_1 = \frac{2}{\sqrt{\beta}\,\sqrt{\gamma-\alpha}+\sqrt{\alpha}\,\sqrt{\gamma-\beta}}$$

$$C_2 = \frac{2}{\sqrt{\beta}\,\sqrt{\gamma+\alpha}+\sqrt{\gamma}\,\sqrt{\beta-\alpha}}$$

$$\kappa = \frac{\sqrt{\beta}\,\sqrt{\gamma-\alpha}-\sqrt{\alpha}\,\sqrt{\gamma-\beta}}{\sqrt{\beta}\,\sqrt{\gamma-\alpha}+\sqrt{\alpha}\,\sqrt{\gamma-\beta}}$$

and

$$\lambda = \frac{\sqrt{\beta}\,\sqrt{\gamma+\alpha}-\sqrt{\gamma}\,\sqrt{\beta-\alpha}}{\sqrt{\beta}\,\sqrt{\gamma+\alpha}+\sqrt{\gamma}\,\sqrt{\beta-\alpha}}. \tag{9}$$

These results are consistent, of course, with the requirement that $OA+BC+EF=AB+CE+FO=0$. The condition that the polygon be bounded by a rectangle is

$$AB+FO=CE=0.$$

This determines the value of d to be

$$d=\frac{2C_1K(\kappa)+C_2K'(\lambda)}{2C_1K(\kappa)-C_2K'(\lambda)}. \tag{10}$$

We see then that the exterior dimensions of the reentrant polygon including the location of strip are given in terms of complete elliptic integrals of the first kind.

It now remains to determine ED from (6)

$$ED=(d-1)\int_\alpha^\delta \frac{du}{R_1(u)} - (d+1)\int_\alpha^\delta \frac{du}{R_2(u)}. \tag{11}$$

In the ω-planes

$$ED=j(d-1)C_1\int_{-1}^{\delta_1}\frac{d\omega}{R_\kappa(\omega)} - (d+1)C_2\int_1^{\delta_2}\frac{d\omega}{R_\lambda(\omega)} \tag{12}$$

where

$$\delta_1=\frac{\sqrt{\alpha\beta}\left(\sqrt{\alpha}\,\sqrt{\gamma-\beta}+\sqrt{\beta}\,\sqrt{\gamma-\alpha}\right)-\left(\sqrt{\alpha}\,\sqrt{\gamma-\alpha}+\sqrt{\beta}\,\sqrt{\gamma-\beta}\right)\delta}{\sqrt{\alpha\beta}\left(\sqrt{\alpha}\,\sqrt{\gamma-\beta}-\sqrt{\beta}\,\sqrt{\gamma-\alpha}\right)+\left(\sqrt{\alpha}\,\sqrt{\gamma-\alpha}-\sqrt{\beta}\,\sqrt{\gamma-\beta}\right)\delta} \tag{13}$$

and

$$\delta_2=\frac{\alpha\beta\gamma-\left(\beta\gamma+\sqrt{\beta\gamma}\,\sqrt{(\alpha+\gamma)(\beta-\alpha)}\right)\delta}{-\alpha\beta\gamma+\left(\beta\gamma-\sqrt{\beta\gamma}\,\sqrt{(\alpha+\gamma)(\beta-\alpha)}\right)\delta}. \tag{14}$$

Now both integrals in (12) can be expressed as elliptic integrals of the first kind in which the lower limit is zero. In the first place

$$\int_{-1}^{\delta_1}\frac{d\omega}{R_\kappa(\omega)}=\int_{-1}^{0}\frac{d\omega}{R_\kappa(\omega)}+\int_0^{\delta_1}\frac{d\omega}{R_\kappa(\omega)}=K(\kappa)+F(\delta_1,\kappa). \tag{15}$$

On the other hand, if in the second integral of (12) $\omega'=\sqrt{\omega^2-1}\,/\lambda'\omega$ with $\lambda'=\sqrt{1-\lambda^2}$ is substituted,

$$\int_1^{\delta_2}\frac{d\omega}{R_\lambda(\omega)}=j\int_0^{\delta_3}\frac{d\omega'}{R_{\lambda'}(\omega')}=jF(\delta_3,\lambda') \tag{16}$$

where $\delta_3=\sqrt{\delta_2^2-1}\,/\lambda'\delta_2$. Finally,

$$ED=j\{(d-1)C_1(K(\kappa)+F(\delta_1,\kappa))-(d+1)C_2F(\delta_3,\lambda')\}. \tag{17}$$

In the summary, the dimensions of the rectangular coaxial structure of Fig. 3, suitably normalized, are given

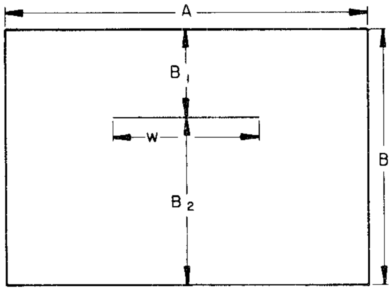

Fig. 3. Coaxial geometry.

in terms of two independent parameters, α and β, chosen so that $0<\alpha<\beta<0.5$ as follows:

$$\begin{aligned} A &= 8K(\kappa)K'(\lambda) \\ B &= 4K(\kappa)K(\lambda)+K'(\kappa)K'(\lambda) \\ B_1 &= 2K'(\kappa)K'(\lambda) \\ B_2 &= 4K(\kappa)K(\lambda)-K'(\kappa)K'(\lambda) \\ W &= 4K(\kappa)F(\delta_3,\lambda')-2K'(\lambda)(K(\kappa)+F(\delta_1,\kappa)) \end{aligned} \tag{18}$$

where

$$\kappa=\frac{\sqrt{1/\alpha-2}-\sqrt{1/\beta-2}}{\sqrt{1/\alpha-2}+\sqrt{1/\beta-2}}$$

$$\lambda=\frac{\sqrt{1+2\alpha}-\sqrt{1-\alpha/\beta}}{\sqrt{1+2\alpha}+\sqrt{1-\alpha/\beta}} \tag{19}$$

and $F(x,k)$ is the incomplete elliptic integral of the first kind of modulus k between the limits 0 and x. Moreover, δ_1 is given by (13) with $\gamma=0.5$ and $\delta=d/(1+d^2)$. Finally, since δ_2 tends to be large,

$$\delta_3=\frac{\sqrt{1-(1/\delta_2)^2}}{\lambda'} \tag{20}$$

where

$$1/\delta_2=\frac{\left(\sqrt{(1+2\alpha)(1-\alpha/\beta)}-1\right)\delta+\alpha}{\left(\sqrt{(1+2\alpha)(1-\alpha/\beta)}+1\right)\delta-\alpha}. \tag{21}$$

IV. The Characteristic Impedances

The characteristic impedance of these coaxial structures is given by η/C where η is the characteristic impedance of the media, $\approx 376.7\ \Omega$ for a vacuum, and C is the geometric capacitance of the structure. Since η depends on the measured velocity of light and is known with only limited accuracy, all further discussion of characteristic impedances will be limited to the evaluation of geometric capacitances.

Clearly the odd-mode characteristic impedance of the structure in Fig. 3 can be obtained by doubling the capacitance of the infinite line segment $FOAB$ with respect to the line segment CDE in the t-plane. Denoting this capacitance by C_0, from the familiar mapping of the upper half of the t-plane into a rectangle, $C_0=2K'(k)/K(k)$ where

$$k^2=\frac{(a'-b')(c'-d')}{(a'-c')(b'-d')}. \tag{22}$$

For this particular case, $a'=b$, $b'=1/b$, $c'=1/a$, and $d'=\infty$. Then

$$k^2=\frac{a}{b}\frac{(1-b^2)}{(1-ab)}. \tag{23}$$

a and b are readily expressed in terms of α and β. In fact,

$$a=\frac{1-\sqrt{1-4\alpha^2}}{2\alpha}$$

and

$$b=\frac{1-\sqrt{1-4\beta^2}}{2\beta} \tag{24}$$

where the minus sign in front of the radical was selected since a and b are less than 1.

The even-mode capacitance C_e in which the outer wall of the rectangular conductor defining B_2 of Fig. 3 is a magnetic wall is obtained by doubling the capacitance of the line segment OAB with respect to the line segment CDE in the t-plane. As above, $C_e=2K'(k)/K(k)$ where k is given by (22), except that now $a'=b$, $b'=1/b$, $c'=1/a$, and $d'=0$. Thus

$$k^2=\frac{1-b^2}{1-ab}. \tag{25}$$

TABLE I
EVEN- AND ODD-MODE CAPACITANCES

B_1/B		W/B 1	25	5	75
5	A/B	1 4347	2 0127	2 8120	3.5714
	C_0	1 9665	2 7015	3 7528	4 7626
	C_e	1 7709	2 2035	2 7533	3 2627
6	A/B	1 6439	2 3067	3 2351	4 1221
	C_0	1 9829	2 7528	3 8522	4 9062
	C_e	1 6194	1 9686	2 4209	2 8445
7	A/B	1 8832	2 6824	3 8284	4 9279
	C_0	2 0792	2 9494	4 2072	5 4153
	C_e	1 4653	1 7547	2 1415	2 5062
8	A/B	2 2217	3 2889	4 8708	6 3967
	C_0	2 3165	3 4261	5 0738	6 6633
	C_e	1 3000	1 5481	1 8906	2 2128
9	A/B	2 9733	4 8342	7 7055	10498
	C_0	3 0034	4 8831	7 7833	10 604
	C_e	1 1077	1 3331	1 6466	1 9372

V. Numerical Results

In Table I the normalized value A/B and the odd- and even-mode capacitances are given for a range of values of W/B and B_1/B.

Since these values can be used to estimate the error in more general approximate solutions of this problem, considerable effort has been made to assure their accuracy. The values of C_0 obtained by the formulas of this paper, for $B_1/B=0.5$, were compared with the values found for the symmetrical case to which it corresponds and for which the general solution is known [4]. In all four cases, the calculated values agreed within one digit in the tenth place. On the other hand, as B_1/B and W/B increase, the accuracy begins to decrease. In the extreme case where $B_1/B=0.9$ and $W/B=0.75$, $\alpha\approx 10^{-6}$, $\lambda'\approx 1-10^{-12}$, and $\delta_3\approx 1-10^{-11}$. As this is continued, the computer will be unable ultimately to distinguish $K(\lambda')$ and $F(\delta_3,\lambda')$ from infinity in the Landen transformations used in these calculations. The fact that α is so small, however, permits one to evaluate these critical integrals, without using the Landen transformations, directly in terms of α with negligible error. Comparison of these accurate values with those obtained from the routines used to find the values given in Table I gave agreement of about one digit in the seventh place for the extreme case when $B_1/B=0.9$ and $W/B=0.75$. On this basis, it is felt that the values in Table I are correct to the nearest digit.

VI. Appendix

If $\kappa=(\sqrt{\beta}\sqrt{\gamma-\alpha}-\sqrt{\alpha}\sqrt{\gamma-\beta})/(\sqrt{\beta}\sqrt{\gamma-\alpha}+\sqrt{\alpha}\sqrt{\gamma-\beta})$, then the linear transformation

$$\omega=\frac{\sqrt{\alpha\beta}\left(\sqrt{\alpha}\sqrt{\gamma-\beta}+\sqrt{\beta}\sqrt{\gamma-\alpha}\right)-\left(\sqrt{\alpha}\sqrt{\gamma-\alpha}+\sqrt{\beta}\sqrt{\gamma-\beta}\right)u}{\sqrt{\alpha\beta}\left(\sqrt{\alpha}\sqrt{\gamma-\beta}-\sqrt{\beta}\sqrt{\gamma-\alpha}\right)+\left(\sqrt{\alpha}\sqrt{\gamma-\alpha}-\sqrt{\beta}\sqrt{\gamma-\beta}\right)u}$$

maps the u-plane into the ω-plane so that $u=0$ transforms into $\omega=-1/\kappa$, $u=\alpha$ goes into $\omega=-1$, $u=\beta$ goes into $\omega=1$, and $u=\gamma$ goes into $\omega=1/\kappa$. Substitution also shows that

$$\frac{du}{\sqrt{u(\alpha-u)(\beta-u)(\gamma-u)}}=\frac{\pm 2j}{\sqrt{\beta}\sqrt{\gamma-\alpha}+\sqrt{\alpha}\sqrt{\gamma-\beta}}\cdot\frac{d\omega}{\sqrt{(1-\omega^2)(1-\kappa^2\omega)}}.$$

The convention regarding the principal values of the integrals requires the selection of the plus sign, and so the value of C_1, used earlier, results.

Similarly, if $\lambda=(\sqrt{\beta}\sqrt{\gamma+\alpha}-\sqrt{\gamma}\sqrt{\beta-\alpha})/(\sqrt{\beta}\sqrt{\gamma+\alpha}+\sqrt{\gamma}\sqrt{\beta-\alpha})$, the linear transformation

$$\omega=\frac{\alpha\beta\gamma-\left(\beta\gamma+\sqrt{\beta\gamma}\sqrt{(\alpha+\gamma)(\beta-\alpha)}\right)u}{-\alpha\beta\gamma+\left(\beta\gamma-\sqrt{\beta\gamma}\sqrt{(\alpha+\gamma)(\beta-\alpha)}\right)u}$$

maps the u-plane into the ω-plane so that $u=-\gamma$ transforms in $\omega=-1/\lambda$, $u=0$ goes into $\omega=-1$, $u=\alpha$ goes into $\omega=1$, and $u=\beta$ goes into $\omega=1/\lambda$. Again, substitution shows that

$$\frac{du}{\sqrt{(\gamma+u)u(\alpha-u)(\beta-u)}}=\frac{\pm 2}{\sqrt{\beta}\sqrt{\alpha+\gamma}+\sqrt{\gamma}\sqrt{\beta-\alpha}}\cdot\frac{d\omega}{\sqrt{(1-\omega^2)(1-\lambda^2\omega^2)}}.$$

Selecting the positive sign for reasons given above, the value of C_2, used earlier, results.

Since $\gamma>\beta>\alpha>0$, it is easily argued that $0<\kappa<1$ as well as $0<\lambda<1$.

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